Existence of Fixed Points for Perov type F-contractions of Hardy-Rogers-Type

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Abstract

In 2012, Wardowski (Fixed Point Theory and Applications **2012** 94 (2012)) introduced the notion of a new type of contraction (nonlinear) namely F-contractions and proved various fixed point theorems for such contractions in the setting of metric spaces. Recently, Altun and Olgun (Journal of Fixed Point Theory and Applications **22** 46 (2020)) initiated the study of F-contractions on vector-valued metric spaces and introduced the concept of Perov type F-contractions. Following this direction of research, in this paper, some new fixed point results for Perov type F-contractions of Hardy-Rogers-Type for self-mappings on spaces equipped with vector-valued metrics are presented. Our results are new and generalize several relevant results in the literature.

Keywords: Metric space, vector-valued metric, F-contraction, fixed point.
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1 Introduction

In 1922, one of the most indispensable results in analysis known as Banach Contraction Principle was formulated by Banach [2] in his doctoral thesis. This principle asserts that every contraction mapping on a complete metric space admits a unique fixed point. This principle is a very effective and popular tool for guaranteeing the existence and uniqueness of the solution of certain problems arising within and beyond mathematics. This principle has inspired many researchers in metric fixed point theory. In the last several decades, several authors generalized and extended the Banach contraction principle in various different ways (see [7, 11, 14, 16] and references therein) by improving contraction conditions, using auxiliary mappings, and enlarging the class of metric spaces for this kind of results.

In 1964, Banach contraction principle was extended for single valued contraction on spaces endowed with vector-valued metrics by Perov [12] and Perov and Kibenko [13]. Lot of Research was done by various authors in this aspect (see, for example, Cvetković and Rakočević [4, 5], Flip and Petruśel [8], Ilić et al. [9] and Vetro and Radenović [18]).

As a new generalization of Banach contraction, Wardowski [19] in 2012 gave the idea of F-contraction. Thereafter, many authors generalized F-contractions in different ways (see [6, 10, 15, 20] and references cited therein). Following this direction of work, Cosentino and Vetro [3] found some fixed point results of Hardy-Rogers-type for self-mappings on complete metric spaces or complete ordered metric spaces in 2014. Very Recently, Altun and Olgun [1] introduced the concept of F-contraction on vector-valued metric spaces.

Inspired by the foregoing observations, some existence and uniqueness fixed point results for F-contractive mappings of Hardy-Rogers-Type on spaces equipped with vector-valued metrics are obtained.

2 Preliminaries

In this section, to make our exposition self-contained, the relevant background material needed to prove our result are presented.

Throughout this paper, $\mathbb{R}_+$ denoted the set of non-negative real numbers, by $\mathbb{R}^m$ the set of $m \times 1$ real matrices, by $\mathbb{R}^m_+$ the set of $m \times 1$ real matrices with non-negative elements, by θ the zero $m \times 1$ matric, by $M_{m\times m}(\mathbb{R}_+)$ the set of all $m \times m$ matrices with non-negative elements, by Θ the zero $m \times m$ matrix, and by I the identity $m \times m$ matrix.

Definition 2.1. *Let A be a nonempty set and $d : A \times A \to \mathbb{R}^m$ be a function. Then d is called a vector-valued metric and (A, d) is said to be vector-valued metric space, if for all $p, q, r \in X$, the following properties are satisfied:*

- $d(p, q) = \theta$ *if and only if* $p = q$,
- $d(p, q) = d(q, p)$,
- $d(p, r) \preceq d(p, q) + d(q, r)$.

In 1964, Perov [12] proved the following theorem in an attempt to generalize the Banach fixed point theorem:

Theorem 2.1. *Let (A, d) be a vector-valued metric space and $F : A \to A$ be a mapping such that there exists a matrix $U \in M_{m,m}(R_+)$, such that:*

$$d(Fa, Fb) \preceq Ud(a, b)$$

for all $a, b \in X$. If U is convergent to zero, then:
(1) F has a unique fixed point in A, say α,
(2) for all $a_0 \in A$, the sequence of successive approximations $\{a_n\}$ defined by $a_n = F^n a_0$ is convergent to α,
(3) the following estimation holds:

$$d(a_n, \alpha) \preceq U^n(I - U)^{-1} d(a_0, Fa_0).$$

In order to generalize Banach contraction principle, Wardowski [19] employed a new type of auxiliary functions as under:

Definition 2.2. *Let $F : \mathbb{R}^m_+ \to \mathbb{R}^m$ be a function. For the sake of completeness, the following conditions are considered:*
(F1) F is strictly increasing in each variable, i.e., for all $\alpha = (\alpha_i)_{i=1}^m$, $\beta = (\beta_i)_{i=1}^m \in \mathbb{R}^m_+$,

such that $\alpha \prec \beta$*, and then,* $F(\alpha) \prec F(\beta)$*,*
(F2) For each sequence $\{a_n\} = (a_n^{(1)}, a_n^{(2)}, ..., a_n^{(m)})$ *of* $\mathbb{R}_+^m$

$$\lim_{n\to\infty} a_n^{(i)} = 0 \text{ if and only if } \lim_{n\to\infty} b_n^{(i)} = -\infty$$

for each $i \in \{1, 2, ..., m\}$*, where:*

$$F((a_n^{(1)}, a_n^{(2)}, ..., a_n^{(m)})) = (b_n^{(1)}, b_n^{(2)}, ..., b_n^{(m)}).$$

(F3) There exists $k \in (0, 1)$*, such that* $\lim_{\alpha_i \to 0^+} \alpha_i^k \beta_i = 0$ *for each* $i \in \{1, 2, ..., m\}$*, where*

$$F((a_1, a_2, ..., a_m)) = (b_1, b_2, ..., b_m).$$

Let $\mathfrak{F}^m$ denote the set of all functions F satisfying $(F1) - (F3)$.
Utilizing above auxiliary functions, the following result was proved in [19]:

Theorem 2.2. *Let* (H, d) *be a complete metric space and* $A : H \to H$*. If there exists* $\tau > 0$ *and* $F \in \mathfrak{F}^m$ *such that*

$$d(Au, Av) > 0 \Rightarrow \tau + F(d(Au, Av)) \leq F(d(u, v)),$$

for all $u, v \in H$*, then* A *possesses a unique fixed point.*

Following this direction of work, Cosentino and Vetro [3] gave some Hardy-Rogers-type fixed point theorems for self-mappings on complete metric spaces or complete ordered metric spaces. The main result of [3] is the following:

Theorem 2.3. *Let* (H, d) *be a complete metric space and let* A *be a self-mapping on* H*. Assume that there exist* $F \in \mathfrak{F}^m$ *and* $\tau \in \mathbb{R}^+$ *such that* A *is an* F*-contraction of Hardy-Rogers-type, that is,*

$$\tau + F(d(Aa, Ab)) \leq F(\alpha.d(a, b) + \beta.d(a, Ta) + \gamma.d(b, Tb) + \delta.d(a, Tb) + L.d(b, Ta)),$$

for all $a, b \in H$*,* $Aa \neq Ab$*, where* $\alpha + \beta + \gamma + 2\delta = 1$*,* $\gamma \neq 1$ *and* $L \geq 0$*. Then* A *has a fixed point. Moreover, if* $\alpha + \delta + L \leq 1$*, then the fixed point of* A *is unique.*

Recently, Altun and Olgun [1] utilized the technique of Wardowski [19] and introduced the following concept of Perov type F-contraction on vector-valued metric spaces.

Definition 2.3. *Let* (H, d) *be a vector-valued metric space and* $A : H \to H$ *be a map. If there exists* $F \in \mathfrak{F}^m$ *and* $\tau = (\tau_i)_{i=1}^m \in \mathbb{R}_+^m$*, such that:*

$$\tau + F(d(Aa, Ab)) \preceq F(d(a, b)), \tag{1}$$

for all $a, b \in H$ *with* $d(Aa, Ab) \succ \theta$*, then* A *is called a Perov type* F*-contraction.*

The main result of [1] is as follows:

Theorem 2.4. *Let* (H, d) *be a complete vector-valued metric space and* $A : H \to H$ *be a Perov type* F*-contraction. Then,* A *has a unique fixed point.*

3 Main Results

By considering the class of F^m, the concept of Perov type F-contraction of Hardy-Rogers type is introduced as follows:

Definition 3.1. *Let (H,d) be a vector-valued metric space and $A : H \to H$ be a map. If there exists $F \in \mathfrak{F}^m$ and $\tau = (\tau)_{i=1}^m \in R_+^m$, such that:*

$$\tau + F(d(Aa, Ab)) \preceq F(\alpha d(a,b) + \beta d(a, Aa) + \gamma d(b, Ab) + \delta d(a, Ab) + Ld(b, Aa)), \qquad (2)$$

for all $a, b \in H$ with $d(Aa, Ab) \succ \theta$, then A is called a Perov type F-contraction of Hardy-Rogers type.

By considering some different function F belonging to F^m in 2, new type of contractions on vector-valued metric spaces are obtained.

The main result of this paper runs as follows.

Theorem 3.1. *Let (H,d) be a complete vector-valued metric space and $A : H \to H$. Assume that there exists $F \in \mathfrak{F}^m$ and $\tau = (\tau)_{i=1}^m \in R_+^m$ such that A is a Perov type F-contraction of Hardy-Rogers-type, that is,*

$$\tau + F(d(Aa, Ab)) \preceq F(\alpha d(a,b) + \beta d(a, Aa) + \gamma d(b, Ab) + \delta d(a, Ab) + Ld(b, Aa)), \qquad (3)$$

for all $a, b \in H$, $Aa \neq Ab$, where $\alpha + \beta + \gamma + 2\delta = 1$, $\gamma \neq 1$ and $L \geq 0$. Then A has a fixed point. Moreover, if $\alpha + \delta + L \leq 1$, then the fixed point of A is unique.

Proof. Let a_0 be an arbitrary point of H. Let $\{a_n\}$ be the Picard sequence with initial point a_0, that is,

$$a_n = A^n a_0 = A a_{n-1},$$

for $n \in \{1, 2, 3, ...\}$. If $a_n = a_{n-1}$ for some n, then a_n is a fixed point of A. Now, let $a_{n+1} \neq a_n$ for every $n \in \{0, 1, 2, ...\}$. Assume that

$$d(a_{n+1}, a_n) = (u_n^{(1)}, u_n^{(2)}, ..., u_n^{(m)}) = u_n \;\; for \;\; n \in \{0, 1, ...\}$$

Then, for all $n \in \{0, 1, 2, ..\}$ and for all $i \in \{1, 2, ..., m\}$, we get

$$\alpha_n^{(i)} > 0$$

In view of (10),

$$\begin{aligned}
\tau + F(u_n) &= \tau + F(d(a_n, a_{n+1})) = \tau + F(d(Ta_{n-1}, Ta_n)) \\
&\preceq F(\alpha d(a_{n-1}, a_n) + \beta d(a_{n-1}, Ta_{n-1}) + \gamma d(a_n, Ta_n) + \delta d(a_{n-1}, Ta_n) + Ld(a_n, Ta_{n-1})) \\
&\preceq F(\alpha(u_{n-1}) + \beta(u_{n-1}) + \gamma(u_n) + \delta(u_{n-1} + u_n)) \\
&= F((\alpha + \beta + \delta).u_{n-1} + (\gamma + \delta).u_n)
\end{aligned}$$

That is,

$$\tau + F((u_n^{(1)}, u_n^{(2)}, ..., u_n^{(m)})) \preceq F((\alpha + \beta + \delta).(u_{n-1}^{(1)}, u_{n-1}^{(2)}, ..., u_{n-1}^{(m)}) + (\gamma + \delta).(u_n^{(1)}, u_n^{(2)}, ..., u_n^{(m)}))$$

Due to the strictly increasing property of the function F,

$$(u_n^{(1)}, u_n^{(2)}, ..., u_n^{(m)}) \prec (\alpha+\beta+\delta).(u_{n-1}^{(1)}, u_{n-1}^{(2)}, ..., u_{n-1}^{(m)}) + (\gamma+\delta).(u_n^{(1)}, u_n^{(2)}, ..., u_n^{(m)}),$$

for all $n \in \mathbb{N}$. The following is obtained from the previous inequality

$$\begin{aligned} u_n^{(i)} &\prec (\alpha+\beta+\delta)u_{n-1}^{(i)} + (\gamma+\delta)u_n^{(i)} \\ (1-\gamma-\delta)u_n^{(i)} &\prec (\alpha+\beta+\delta)u_{n-1}^{(i)} \\ u_n^{(i)} &\prec \frac{\alpha+\beta+\delta}{1-\gamma-\delta}u_{n-1}^{(i)} \\ u_n^{(i)} &\prec u_{n-1}^{(i)}, \end{aligned}$$

for all $i \in \{1, 2, ..., m\}$ and $n \in \mathbb{N}$. Hence,

$$(\tau_i)_{i=1}^m + F(u_n^{(i)}) \preceq F(u_{n-1}^{(i)}),$$

for all $i \in \{1, 2, ..., m\}$ and $n \in \mathbb{N}$. This gives

$$\begin{aligned} F(u_n^{(i)}) &\preceq F(u_{n-1}^{(i)}) - (\tau_i)_{i=1}^m \\ &\preceq F(u_0^{(i)}) - n.(\tau_i)_{i=1}^m, \end{aligned} \tag{4}$$

for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} F(u_n^{(i)}) = -\infty$. In view of (F2)

$$\lim_{n\to\infty} u_n^{(i)} = 0,$$

for all $i \in \{1, 2, ..., m\}$. Now, from (F3), there exists $k \in (0, 1)$, such that:

$$\lim_{n\to\infty} (u_n^{(i)})^k F(u_n^{(i)}) = 0$$

for all $i \in \{1, 2, ..., m\}$. From (4),

$$(u_n^{(i)})^k F(u_n^{(i)}) - (u_n^{(i)})^k F(u_0^{(i)}) \preceq -(u_n^{(i)})^k n(\tau_i)_{i=1}^m \preceq 0. \tag{5}$$

Letting $n \to \infty$ in (5),

$$\lim_{n\to\infty} n(u_n^{(i)})^k = 0 \tag{6}$$

for all $i \in \{1, 2, ..., m\}$. From (6), we have $n^{(i)} \in \{1, 2, ...\}$, such that $n(u_n^{(i)})^k \leq 1$ for all $n \geq n^{(i)}$. So,

$$a_n^{(i)} \leq \frac{1}{n^{\frac{1}{k}}}, \tag{7}$$

for all $n \geq n_0 = \max\{n^{(i)} : i \in \{1, 2, ..., m\}\}$:.
Consider $p, q \in \mathbb{N}$, such that $p > q \geq n_0$. Now, from (7) and the triangular inequality

property for the vector-valued metric, following is obtained

$$\begin{aligned} d(a_q, a_p) &\preceq d(a_q, a_{q+1}) + d(a_{q+1}, a_{q+2}) + ... + d(a_{p-1}, a_p) \\ &= a_q + a_{q+1} + ... + a_{p-1} \\ &= (a_q^{(i)})_{i=1}^m + (a_{q+1}^{(i)})_{i=1}^m + ... + (a_{p-1}^{(i)})_{i=1}^m \\ &= \left(\sum_{r=q}^{p-1} a_r^{(i)}\right)_{i=1}^m \\ &\preceq \left(\sum_{r=q}^{\infty} a_r^{(i)}\right)_{i=1}^m \\ &\preceq \left(\sum_{r=q}^{\infty} \frac{1}{r^{1/k}}\right)_{i=1}^m \end{aligned}$$

Due to the fact that the series $\sum_{r=1}^{\infty} \frac{1}{r^{1/k}}$ is convergent, so

$$d(a_q, a_p) \to \theta \text{ as } q \to \infty.$$

This shows that $\{a_n\}$ is a Cauchy sequence in (H, d). As H is a complete vector-valued metric space, the sequence $\{a_n\}$ converges to some point $\mu \in H$, that is, $\lim_{n\to\infty} a_n = \mu$. On the other hand from (F1) and (10),

$$d(Aa, Ab) \preceq \alpha d(a, b) + \beta d(a, Aa) + \gamma d(b, Ab) + \delta d(a, Ab) + Ld(b, Aa),$$

for all $a, b \in H$, $Aa \neq Ab$. Therefore,

$$d(Aa_n, A\mu) \preceq \alpha d(z_n, \mu) + \beta d(a_n, Aa_n) + \gamma d(\mu, A\mu) + \delta d(a_n, A\mu) + Ld(\mu, Aa_n);$$

that is:

$$\begin{aligned} d(\mu, A\mu) &\preceq \gamma d(\mu, A\mu) + \delta d(\mu, A\mu) \\ d(\mu, A\mu) &\preceq (\gamma + \delta).d(z, A\mu) \prec d(\mu, A\mu), \end{aligned}$$

which is a contradiction and hence $A\mu = \mu$.

Now, to prove the uniqueness of the fixed point assume that $\lambda \in H$ is another fixed point of A, different from μ. This means that $d(\mu, \lambda) > 0$. Put $a = \mu$ and $b = \lambda$ in (10),

$$\begin{aligned} \tau + F(d(\mu, \lambda)) &= \tau + F(d(A\mu, A\lambda)) \\ &\preceq F(\alpha d(\mu, \lambda) + \beta d(\mu, A\mu) + \gamma d(\lambda, A\lambda) + \delta d(\mu, A\lambda) + Ld(\lambda, A\mu)), \\ &\preceq F(\alpha d(\mu, \lambda) + \delta d(\mu, \lambda) + Ld(\lambda, \mu)) \\ &= F((\alpha + \delta + L)d(\mu, \lambda)), \end{aligned}$$

which is a contradiction since $\alpha + \delta + L \leq 1$, and hence $\mu = \lambda$. □

As a first corollary of Theorem 3.1, taking $\alpha = 1$ and $\beta = \gamma = \delta = L = 0$, the main theorem of Altun and Olgun [1] is obtained.

Corollary 3.2. *Let* (H, d) *be a complete vector-valued metric space and* $A : H \to H$*. Assume that there exists* $F \in \mathfrak{F}^m$ *and* $\tau = (\tau)_{i=1}^m \in \mathbb{R}_+^m$ *such that*

$$\tau + F(d(Aa, Ab)) \preceq F(\beta d(a, Aa) + \gamma d(b, Ab)), \tag{8}$$

for all $a, b \in H$*,* $Aa \neq Ab$*, where* $\beta + \gamma = 1$*,* $\gamma \neq 1$*. Then* A *has a unique fixed point.*

Proof. By putting $\alpha = \delta = L = 0$ and $\beta + \gamma = 1$ and $\beta \neq 0$, the proof is obtained. □

Corollary 3.3. *Let* (H, d) *be a complete vector-valued metric space and* $A : H \to H$*. Assume that there exists* $F \in \mathfrak{F}^m$ *and* $\tau \in \mathbb{R}_+^m$ *such that*

$$\tau + F(d(Aa, Ab)) \preceq F\left(\frac{1}{2}d(a, Ab) + Ld(b, Aa)\right), \tag{9}$$

for all $a, b \in H$*,* $Aa \neq Ab$*. Then* A *has a fixed point in* H*. If* $L \leq 1/2$*, then we have unique fixed point of* A*.*

Proof. By putting $\alpha = \beta = \gamma = 0$ and $\delta = \frac{1}{2}$, the proof is obtained. □

Corollary 3.4. *Let* (H, d) *be a complete vector-valued metric space and* $A : H \to H$*. Assume that there exists* $F \in \mathfrak{F}^m$ *and* $\tau \in \mathbb{R}_+^m$ *such that*

$$\tau + F(d(Aa, Ab)) \preceq F\left(\alpha d(a, b) + \beta d(a, Aa) + \gamma d(b, Ab)\right), \tag{10}$$

for all $a, b \in H$*,* $Aa \neq Ab$*, where* $\alpha + \beta + \gamma = 1$*,* $\gamma \neq 1$*. Then* A *has a unique fixed point in* H*.*

Proof. By putting $\delta = L = 0$, the proof is obtained. □

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